Exercise 21

(a) If \$3000 is invested at 5% interest, find the value of the investment at the end of 5 years if the interest is compounded (i) annually, (ii) semiannually, (iii) monthly, (iv) weekly, (v) daily, and (vi) continuously.

(b) If $A(t)$ is the amount of the investment at time t for the case of continuous compounding, write a differential equation and an initial condition satisfied by $A(t)$.

Solution

Part (a)

The value of an investment that has compounding interest (n times per year with an interest rate r) is

$$A(t) = A_0\left(1 + \frac{r}{n}\right)^{nt}.$$

The amount owed back at the end of 5 years if the interest is compounded annually is

$$A(5) = 3000\left(1 + \frac{0.05}{1}\right)^{(1)5} \approx \$3828.84.$$

The amount owed back at the end of 5 years if the interest is compounded semiannually is

$$A(5) = 3000\left(1 + \frac{0.05}{2}\right)^{(2)5} \approx \$3840.25.$$

The amount owed back at the end of 5 years if the interest is compounded monthly is

$$A(5) = 3000\left(1 + \frac{0.05}{12}\right)^{(12)5} \approx \$3850.08.$$

The amount owed back at the end of 5 years if the interest is compounded weekly is

$$A(5) = 3000\left(1 + \frac{0.05}{\frac{365}{7}}\right)^{\left(\frac{365}{7}\right)5} \approx \$3851.61.$$

The amount owed back at the end of 5 years if the interest is compounded daily is

$$A(5) = 3000\left(1 + \frac{0.05}{365}\right)^{(365)5} \approx \$3852.01.$$

The amount owed back at the end of 5 years if the interest is compounded continuously is

$$A(5) = \lim_{n\to\infty} 3000\left(1 + \frac{0.05}{n}\right)^{(n)5} = 3000e^{0.05(5)} \approx \$3852.08.$$

Part (b)

If the interest is compounded continuously, the amount owed back after t years is

$$A(t) = A_0 e^{rt},$$

where A_0 is the amount borrowed and r is the interest rate.

$$A(t) = 3000e^{0.05t}$$

Take the derivative and write it in terms of $A(t)$ to get the differential equation.

$$\frac{dA}{dt} = \frac{d}{dt}(3000e^{0.05t}) = 3000(0.05e^{0.05t}) = 0.05(3000e^{0.05t}) = 0.05A(t)$$

The initial condition associated with it is

$$A(0) = 3000e^{0.05(0)} = 3000.$$